Relativistic fluid spheres with particular application in cosmology and gravitational collapse

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Abstract

A spherically symmetric comoving fluid solution of Einstein's equations is adapted for cosmological application by extending the geometry of standard FRW cosmology using a generalised curvature term. The resulting model retains many of the known cosmological properties including homogeneity of energy density, its relationship with internal pressure including equations of state, although in each case they have a generalised structure. It is shown that the adapted model does not require the inclusion of the arbitrary cosmological constant and the vacuum energy solution is discussed in its absence. The Hubble constant and deceleration parameter are also shown to have a form which characterises the modified geometry of the new model. These forms are calculated using current observational data and show how the standard cosmological geometry can be amended in a way which is consistent with an observed flat curvature and a decelerating universe. Finally the solution is also considered in the context of gravitational collapse where it is shown how fluids spheres obeying a central equation of state can be matched to empty spacetime.

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1 Introduction

Spherically symmetric fluid solutions of Einstein's equations have been discussed in depth by many authors for many reasons which include their important wide

ranging applications to cosmology and astrophysics. General solutions have often been obtained using a metric in isotropic form together with a comoving system of coordinates. From the point of view this paper the approach originally described by Kustaanheimo and Qvist [1], has been invaluable although although there was an earlier particular solution presented by McVittie [2]. Subsequently there have other investigations which often involved Lie symmetry approaches for example Stephani [3], Stephani, Wolf [4] and Wiltshire [5]. In addition have been many comprehensive studies of their mathematical, physical properties and their various interrelationships by for example Nariai [6] ,Chakravarty [7], McVittie [8, 9], Srivastava [10], Sussman [11, 12, 13], McVittie [8], Knutsen [14]. Furthermore the host of solutions have been summarised by Krasinski [15] and also Stephani *et al* [16].

Nonetheless there remain many outstanding problems associated with the physical applicability of these solutions. For example in cosmology the simplest spherically symmetric solution gives rise to the Friedmann, Robertson, Walker (FRW) standard cosmological models which have recently required significant adaptation to describe emerging observational data. The evidence of very early epoch, inflation, the notion of vacuum energy and a negative deceleration parameter in a more or less flat universe (based upon type 1a supernova surveys Perlmutter, Riess, Schmidt, Garnavich & coworkers [17, 18, 19, 20, 21]) have created many challenges for the standard model and general relativity. This has required the reintroduction of the cosmological constant and the resulting concordance or ΛCDM model to reconcile observation with theory by means of large proportions of dark energy $\Omega_\Lambda = 0.7$ and cold dark matter $\Omega_M = 0.23$. Recent developments Turner & Reiss [22], Virey [23] show that even this may not be sufficient, in fact a bimodal model may be required to describe past deceleration as well as recent acceleration of the universe. In the following it will be seen that there is a generalisation of FRW cosmology which overcomes many of these difficulties without the introduction of a cosmological constant.

Moreover this solution will be shown to simplify many difficulties in the description of problems of gravitational collapse where in practice it is extremely difficult in practice to match a fluid sphere to Schwarzschild empty space-time even though the theory is well known, Bonnor & Vickers [24]. This is especially true in cases where it is necessary to incorporate equation of state between pressure and energy density $p = p(\rho)$ for example at the centre of the sphere. For this reason the Oppenheimer Snyder model [25] essentially a zero pressure FRW solution has often been employed in matching problems as has been the case in discussions of a potential source for gravitational waves Babak & Glampedakis [26] or a matching problem in first order rotation Kegeles [27].

This paper is organised in the following way. Einstein's equations in the context of general spherically symmetric fluid spheres are introduced in section 1 with a focus on the particular solution to be discussed. The basic properties of the solution are presented in section 3 in relation to the FRW cosmological models and the cosmological constant. The solution is then discussed in the context of vacuum energy in section 4 whilst section 5 focusses on equations of state and problems of gravitational collapse. Finally in section 6 the solution

is discussed in terms of the Hubble constant, deceleration parameter and some current data.

2   Fluid spheres, isotropy, preliminary equations

It is the intention here to consider the isotropic coordinate system for which

$$ds^2 = e^{2\lambda}dt^2 - e^{2\mu}\left(dr^2 + r^2 d\Omega^2\right) \qquad\qquad d\Omega^2 = d\theta^2 + \sin^2\left(\theta\right)d\phi^2 \qquad (1)$$

where $\lambda = \lambda\left(r,t\right)$, $\mu = \mu\left(r,t\right)$. In addition Einstein's field equations will be taken to be

$$G_k^i = -8\pi T_k^i \qquad\qquad T_k^i = \left(\rho + p\right)u^i u_k - \delta_k^i p \qquad (2)$$

where G_k^i is the Einstein tensor and T_k^i is the energy momentum tensor of the fluid sphere. A comoving coordinate system will assumed so that the components u^i of the velocity four vector satisfy $u^i u_i = 1$ with $u^1 = 0$ and $u^2 = 0 = u^3$. The pressure p, energy density ρ and mass function m, see for example, Misner and Sharp [28], Cahill and McVittie [29] may be calculated using:

$$8\pi p = G_2^2 \qquad 8\pi\rho = -G_4^4 \qquad m = \frac{re^\mu}{2}\left\{1 + e^{2(\mu-\lambda)}r^2\mu_t^2 - \left(1 + r\mu_r\right)^2\right\} \qquad (3)$$

where the suffix r, t indicates a partial derivative. In this system Einstein's field equations satisfy the isotropy condition in the form

$$G_1^4 = 0 = G_4^1 \qquad\qquad G_2^2 - G_1^1 = 0 \qquad (4)$$

and so

$$\mu_{rt} - \lambda_r\mu_t = 0 \qquad (5)$$

and

$$\mu_{rr} + \lambda_{rr} + \lambda_r^2 - \mu_r^2 - 2\lambda_r\mu_r - \frac{\left(\mu_r + \lambda_r\right)}{r} = 0 \qquad (6)$$

Using the approach of Kustaanheimo and Qvist [1] the solutions of (5) and (6) may be expressed in terms of the function $L = L\left(x,t\right)$ where

$$L = e^{-\mu} \qquad x = r^2 \qquad e^\lambda = A\left(t\right)\mu_t = -A\left(t\right)\frac{L_t}{L} \qquad (7)$$

In this notation (5) and (6) are simultaneously satisfied by

$$L_{xx} = L^2 F\left(x\right) \qquad (8)$$

In general this equation has been studied for the mathematically tractable cases $F\left(x\right) = \left(ax^2 + bx + c\right)^{-5/2}$ often leading to solutions involving elliptic functions, see for example Stephani *et al* [16] whilst the much simpler form

$$L\left(x,t\right) = f(t)x + g(t) \qquad\qquad F\left(x\right) = 0 \qquad (9)$$

3

leads to the FRW cosmological models and also the Oppenheimer- Synder model for gravitational collapse [25]. However the FRW solution

$$L\left(x,t\right) = \frac{1}{R\left(t\right)}\left(1 + \frac{kx}{4}\right) \qquad F\left(x\right) = 0 \qquad (10)$$

is only a subcase of the more general class given by (9) which will be the primary subject of study below. It will be shown how this generalised class leads to a new class of fluid spheres possessing uniform energy density with the prospect of extending the class of FRW cosmologies and also fluid sources for gravitational collapse.

3   The solution and basic properties

In the context here it helpful to defines the function $S\left(r,t\right)$ for which

$$S = \frac{rR}{1 + \frac{\sigma R^2 r^2}{4}} \qquad (11)$$

and a solution of Einstein's equations having the general structure of (9) in the form:

$$e^{-\mu} \equiv \frac{r}{S} = \frac{\left(1 + \frac{\sigma R^2 r^2}{4}\right)}{R} \qquad (12)$$

so that by (7)

$$e^{\lambda} = \frac{S_t}{S}a^{-\frac{1}{2}} = \left(\frac{R_t}{R} - \frac{r^2 R\left(2\sigma R_t + \sigma_t R\right)}{4\left(1 + \frac{\sigma R^2 r^2}{4}\right)}\right)a^{-\frac{1}{2}} \qquad (13)$$

where $R = R\left(t\right)$, $\sigma = \sigma\left(t\right)$ and $a = a\left(t\right)$ and satisfy (5) and (6) by direct substitution. The Ricci scalar curvature for the 3 dimensional space in (1) with (12) has the value 6σ so that the solution is closed when $\sigma > 0$, flat when $\sigma = 0$ or open when $\sigma < 0$.

Also note that the FRW solution is obtained as a subcase of (1) with (12) and (13) by writing

$$\sigma = \frac{k}{R^2} \qquad a = \frac{S_t^2}{S^2} = \frac{R_t^2}{R^2} \qquad (14)$$

The energy density $\rho = \rho\left(t\right)$ from (3) is:

$$8\pi\rho = 3\left(a + \sigma\right) \qquad (15)$$

It is a purely time dependent equation and is a generalisation of the normal Friedmann equation when the condition (14) is also to be included. Equation

4

(15) demonstrates that the energy density is a composite of two functions $a(t)$ and $\sigma(t)$ neither of which have any *a priori* dependence of the scale factor $R(t)$. However the pressure $p = p(r, t)$ calculated using (3) is in general a function of both r and $R(t)$ and is given through

$$8\pi (p + \rho) = \frac{(a_t + \sigma_t) R (4 + \sigma r^2 R^2)}{(r^2 \sigma R^2 R_t - 4 R_t + r^2 \sigma_t R^3)} \tag{16}$$

Equations (15) and (16) represent the generalised form of the two Friedmann equations which form the basis of standard cosmology. Subsequent analysis in this paper will be based upon these generalised forms. It can be shown that (16) may be cast into a more familiar form to cosmologists by combining it with $S(t)$ defined in (11) and the derivative of (15) to give

$$p \frac{\partial S^3}{\partial t} = -\frac{\partial (\rho S^3)}{\partial t} \qquad \Longrightarrow \qquad \rho_t = -\frac{3 (p + \rho) S_t}{S} \tag{17}$$

Notice that when either $r = 0$ or alternatively $\sigma = k/R^2$ this reduces to standard form

$$\rho_t = -\frac{3 (p + \rho) R_t}{R} \tag{18}$$

In the addition the mass function has the form

$$m = \frac{4\pi \rho S^3}{3} = \frac{4\pi \rho (Rr)^3}{3} \left(1 + \frac{\sigma R^2 r^2}{4}\right)^{-3} \tag{19}$$

also part of standard cosmology and which also be used in cases of gravitational collapse to define a fluid sphere boundary.

As can be seen from equation (13) it is not possible in general to choose a function $a(t)$ such that $e^\lambda = 1$ for all values of r and so define a comprehensive proper time variable, that is with the exception of the FRW cases given by (14). However it is possible for an observer at $r = \kappa$ to define a *local* proper time variable by setting

$$S(\kappa, t) = K(t) \qquad \Longrightarrow \qquad S(0, t) = R(t) \tag{20}$$

and using (13)

$$a(t) = \frac{K_t^2}{K^2} = \left(\frac{R_t}{R} - \frac{\kappa^2 R (2\sigma R_t + \sigma_t R)}{4 \left(1 + \frac{\sigma R^2 k^2}{4}\right)}\right)^2 \tag{21}$$

In this way (15) becomes

$$8\pi \rho = 3 \left(\frac{K_t^2}{K^2} + \sigma\right) \tag{22}$$

whilst the pressure from (17) now satisfies

$$p\frac{dK^3}{dt} = -\frac{d\left(\rho K^3\right)}{dt} \qquad or \qquad \rho_t = -\frac{3\left(p+\rho\right)K_t}{K} \tag{23}$$

Without loss of generality in the following it will be assumed that local propertime is defined at the coordinate centre $r = 0$ so that:

$$8\pi\rho = 3\left(\frac{R_t^2}{R^2} + \sigma\right) \tag{24}$$

whilst from (23)

$$8\pi p = -\frac{2R_{tt}}{R} - \frac{R_t^2}{R^2} - \frac{\sigma_t R}{R_t} - 3\sigma \tag{25}$$

Clearly in the particular case when $\sigma = k/R^2$ then (24) and (25) give rise to the Friedmann equations. Note also that on writing

$$\sigma = \bar{\sigma} + \frac{\chi}{3} \tag{26}$$

where $\bar{\sigma} = \bar{\sigma}\left(t\right)$ and χ is a constant then (24) and (25) become respectively:

$$8\pi\rho = 3\left(\frac{R_t^2}{R^2} + \bar{\sigma}\right) + \chi \qquad 8\pi p = -\frac{2R_{tt}}{R} - \frac{R_t^2}{R^2} - \frac{\bar{\sigma}_t R}{R_t} - 3\bar{\sigma} - \chi \tag{27}$$

Thus the choice of σ at (26) results in the systematic inclusion of a constant, χ equivalent to the cosmological constant Λ $(\equiv -\chi)$ normally associated with Einstein's equations. However, in terms of cosmological applications it is better to consider equations (24) and (25) in their full generality as σ is *a priori* an undefined function of t and so has the capacity to act as a 'variable cosmological constant'. For example, Perivolaropoulos [30] describes the *cosmological constant problem* whereby the cosmological constant is considered to have had relatively large value during the early period of rapid inflation much larger than during the current epoch.

Finally note that the metric (1) with (12) and (13) remains invariant under the transformation

$$r = \frac{1}{\bar{r}} \qquad R = \frac{1}{\sigma\bar{R}} \tag{28}$$

where $\bar{R} = \bar{R}\left(t\right)$. In particular the transformation gives to

$$ds^2 = e^{2\bar{\lambda}}dt^2 - e^{2\bar{\mu}}\left(d\bar{r}^2 + \bar{r}^2 d\Omega^2\right) \tag{29}$$

where

$$e^{-\bar{\mu}} = \frac{\left(1 + \frac{\sigma\bar{R}^2 r^2}{4}\right)}{\bar{R}} \qquad e^{\bar{\lambda}} = \left(\frac{\bar{R}_t}{\bar{R}} - \frac{r^2\bar{R}\left(2\sigma\bar{R}_t + \sigma_t\bar{R}\right)}{4\left(1 + \frac{\sigma\bar{R}^2 r^2}{4}\right)}\right)a^{-\frac{1}{2}} \tag{30}$$

Thus the properties of the energy density and pressure found at (r, t) are replicated at $(\bar{r}, t)$ using (28). In particular the properties found at $r = 0$ are replicated at infinity.

4   Vacuum energy and equation of state $p = -\rho$

Consider now the case in (26) when

$$\bar{\sigma} = -\frac{R_t^2}{R^2} \qquad\qquad \sigma = \bar{\sigma} + \frac{\chi}{3} \tag{31}$$

so that equation (27) reduces to the vacuum energy equation of state (see for example, Peacock [31])

$$8\pi\rho = \chi \qquad\qquad p = -\rho \tag{32}$$

The full solution is now given by equations (12) and (13)

$$e^{-\mu} = \frac{1}{R}\left\{1 + \frac{r^2 R^2}{4}\left(\frac{\chi}{3} - \frac{R_t^2}{R^2}\right)\right\} \qquad e^{\lambda} = 1 - \frac{r^2 R^2 \left(\frac{\chi}{3} - \frac{R_{tt}}{R}\right)}{2\left\{1 + r^2 R^2 \left(\frac{\chi}{3} - \frac{R_t^2}{R^2}\right)\right\}} \tag{33}$$

If proper time is employed then on writing $\alpha^2 = \chi/3$ with (31) then

$$R = e^{\alpha t} \qquad\qquad e^{-\mu} = \frac{1}{R} \qquad\qquad e^{\lambda} = 1 \tag{34}$$

and also

$$\sigma = 0 \qquad\qquad m = \frac{cr^3 e^{3\alpha t}}{2} \tag{35}$$

When $\chi = 0$ then from (19) a Minkowski spacetime is obtained represented by

$$e^{-\mu} = \frac{1}{R}\left\{1 - \frac{r^2 R_t^2}{4}\right\} \qquad\qquad e^{\lambda} = 1 + \frac{r^2 R R_{tt}}{2\left\{1 - \frac{r^2 R_t^2}{4}\right\}} \tag{36}$$

Thus using proper time

$$R = 1 + \alpha t \qquad\qquad e^{-\mu} = \frac{1}{R}\left\{1 - \frac{\alpha^2 r^2}{4}\right\} \qquad\qquad e^{\lambda} = 1 \tag{37}$$

with

$$\sigma = -\frac{\alpha^2}{(1 + \alpha t)^2} \qquad\qquad m = 0 \tag{38}$$

Equation (37) is of a form described by Milne [32] and Peacock [31].

5   Equations of state

5.1   Cosmological application

Consider the cosmological case so that there is no fluid boundary and where it will be assumed that a fluid equation of state $p = p(\rho)$ exists at $r = 0$. In

addition local proper time is used so that so that from equation (21)

$$a = \frac{R_t^2}{R^2} \tag{39}$$

With $K = R$, equation (23) may be solved for particular $p = p(\rho)$ and the results which have a familiar form are summarised in Table 1.

Equation of state (at $r = 0$)	Pressure p	Energy density ρ (α is constant)
Adiabatic $(\gamma \neq 1)$	$p = N\rho^\gamma$	$\rho = \left(\alpha R^{3(\gamma-1)} - N\right)^{\frac{1}{1-\gamma}}$
Linear general (all values of n)	$p = (n-1)\rho$	$\rho = \dfrac{\alpha}{R^{3n}}$
Dust $(n = 1)$	$p = 0$	$\rho = \dfrac{\alpha}{R^3}$
Radiation domination $(n = 4/3)$	$p = \dfrac{\rho}{3}$	$\rho = \dfrac{\alpha}{R^4}$
Vacuum energy $(n = 0)$	$p = -\rho$	$\rho = \alpha$

Table 1

Thus the function $\sigma(t)$ can then be determined from equation (24) so that

$$\sigma = \frac{8\pi\rho}{3} - \frac{R_t^2}{R^2} \tag{40}$$

This equation determines the function $\sigma(t)$. In FRW cosmology $\sigma = k/R^2$ then equation (40) is a differential equation that is solved to determine $R = R(t)$.

5.2 Gravitational collapse application

In cases of gravitational collapse suppose that a fluid sphere has an equation of state $p = p(\rho)$ defined at the centre $r = 0$ which from equation (17) gives results which are identical to those described in Table 1. In this context the adiabatic equation of state is particularly relevant to polytropic stars as has been described by Weinberg [33]. However it is not assumed that local proper time is used at $r = 0$ and so equation (21) does not hold. However it is supposed that the fluid body has a well defined boundary $r = b$ which matches the Scharzschild vacuum solution and where the boundary value of the pressure must be zero.

For zero pressure at $r = b$ it is well known, see for example, Bonnor & Vickers [24] or Cahill and McVittie [29] that the mass function (19) is a constant $m = M$. Thus

$$M = \frac{4\pi\rho S^3}{3} = \frac{4\pi\rho B^3}{3} \tag{41}$$

where

$$B = B(t) = \frac{Rb}{1 + \frac{\sigma R^2 b^2}{4}} \qquad (42)$$

so that

$$\frac{d}{dt}\left(\rho B^3\right) = 0 \qquad \Longrightarrow \qquad B = \frac{\kappa}{\rho^{1/3}} \qquad (43)$$

Equations (42) together with (43) may used to show that:

$$\sigma = -\frac{4}{R^2 b^2 \kappa}\left(\kappa + Rb\rho^{1/3}\right) \qquad (44)$$

whilst from equation (15)

$$a = \frac{8\pi\rho}{3} + \frac{4}{R^2 b^2 \kappa}\left(\kappa + Rb\rho^{1/3}\right) \qquad (45)$$

It follows for (17) with (43) that

$$p\frac{\partial S^3}{\partial t} = -\frac{\partial\left(\rho\left[S^3 - B^3\right]\right)}{\partial t} \qquad (46)$$

which gives the expression for pressure such that $p(b,t) = 0$ as required.

Finally note that each of these fluid spheres when endowed with first order rotation in terms of a rotation parameter may also be matched to empty spacetime using a general result given by Wiltshire [34].

6   Hubble constant and deceleration parameter

A further understanding of the geometry of this solution and its relation to current cosmological observational data may be obtained by determining the Hubble constant and deceleration parameter for the solution.

The null geodesic equation for an inward travelling photon as described by an observer at $(r,t) = (0,t_0)$ using proper time is

$$\frac{dr}{dt} = -e^{\lambda-\mu} \qquad \text{with} \qquad e^\lambda = 1 \qquad (47)$$

With the notation that

$$t = t_0 - \tau \qquad (48)$$

where τ is the travel time of the photon and using (12) and (13) equation (47) expressed in terms of travel time τ becomes:

$$\frac{dr}{d\tau} = \frac{1 + r^2 \sigma R^2}{R} \qquad e^\lambda = \left(\frac{R_t}{R} - \frac{r^2 R\left(2\sigma R_t + \sigma_t R\right)}{4\left(1 + \frac{\sigma R^2 r^2}{4}\right)}\right) a^{-\frac{1}{2}} = 1 \qquad (49)$$

9

The equations (49) are solved upto and including second order terms in τ by writing

$$R = R_0 - \dot{R}_0\tau + \frac{\ddot{R}_0}{2}\tau^2 \qquad \sigma = \sigma_0 - \dot{\sigma}_0\tau + \frac{\ddot{\sigma}_0}{2}\tau^2$$

$$a = a_0 - \dot{a}_0\tau + \frac{\ddot{a}_0}{2}\tau^2 \tag{50}$$

where for example $\dot{R}$ means R_t evaluated at $t = t_0$. In this way the solution the first of (49) and for example the redshift z can be calculated in the usual way

$$r(\tau) R_0 = \tau + \frac{H}{2}\tau^2 \qquad \text{and} \qquad z = H\tau + H^2\frac{(q+2)}{2}\tau^2 \tag{51}$$

where the Hubble constant and deceleration parameter are

$$H = \frac{\dot{R}_0}{R_0} \qquad\qquad q = -\frac{R_0\ddot{R}_0}{\dot{R}_0^2} \tag{52}$$

The condition that $e^\lambda = 1$ is given by

$$a_0 = \frac{\dot{R}_0^2}{R_0^2} \qquad\qquad \dot{a}_0 = \frac{2\dot{R}_0\ddot{R}_0}{R_0^2} - 2\frac{\dot{R}_0^3}{R_0^3} \tag{53}$$

and is essentially the statement that $a = (R_t/R)^2$ upto and including first order terms in τ.

With this notation the first of (53) with (15) and (52) gives

$$H^2 = \frac{\dot{R}_0^2}{R_0^2} = \frac{8\pi\rho_0}{3} - \sigma_0 \tag{54}$$

whilst the second of (53) with (52) together with (18) results in

$$q = -\frac{R_0\ddot{R}_0}{\dot{R}_0^2} = \frac{\dot{\sigma}_0}{2\left(\dfrac{8\pi\rho_0}{3} - \sigma_0\right)^{3/2}} + \frac{4\pi(p_0 + \rho_0)}{\left(\dfrac{8\pi\rho_0}{3} - \sigma_0\right)} - 1 \tag{55}$$

So when $p = (n-1)\rho$ then equations (54) and (55) may be taken to give

$$q = \frac{\dot{\sigma}_0}{2\left(\dfrac{8\pi\rho_0}{3} - \sigma_0\right)^{3/2}} + \frac{4\pi n\rho_0}{\left(\dfrac{8\pi\rho_0}{3} - \sigma_0\right)} - 1 \tag{56}$$

Thus in the case when $\sigma = 0$ for all t then

$$q = \frac{3n}{2} - 1 \tag{57}$$

as expected. In addition for the case of vacuum energy $n = 0$ and σ is constant then $q = -1$ again as required.

10

In the light of recent observations for example WMAP [35] evidence from the cosmic microwave background suggests that the universe is essentially flat so that in the context of this model $\sigma_0 = 0$ so that

$$H^2 = \frac{8\pi\rho_0}{3} \tag{58}$$

as is normally calculated. In addition however observations of distant type Ia supernovae suggest that $q < 0$ and that the universe is decelerating rapidly. In this model (56) can be rearranged for $\dot\sigma_0$ incorporating $\sigma_0 = 0$ so that:

$$\dot\sigma_0 = \left(\frac{8\pi\rho_0}{3}\right)^{3/2}(2q - 3n + 2) \tag{59}$$

With equation (58) this is also

$$\dot\sigma_0 = H^3(2q - 3n + 2) \tag{60}$$

Thus for example a matter dominated universe, $n = 1$ with $q = -1/2$ gives

$$\dot\sigma_0 = -2H^3 \tag{61}$$

Hence it has be shown how the observed data can be used to determine the geometric parameters $[\sigma_0, \dot\sigma_0]$ which characterise the difference between the geometry of the current model with that of standard FRW cosmology.

7   Conclusion

In this paper the primary focus has been on consideration of an extended version of the FRW solution which can be characterised by a Ricci curvature scalar having a value $6\sigma(t)$ which is independent of the scale factor $R(t)$. Moreover the purely time dependent energy density consists of the sum of two terms which from a geometrically point of view are the curvature and also a function $a(t)$ that defines the nature of the time coordinate, for example proper time. This is an extended form of the Friedmann equation and contrasts with the FRW solution for which the curvature and also the energy density depend closely on the scale factor. It has further been shown how expression for internal pressure now a function of (r, t) gives rise to an equation which is closely analogous to the second Friedmann equation which expresses the acceleration of the scale factor in the standard model. The generalised equations are then shown by a translation of the curvature function to contain naturally a constant that can be interpreted as a cosmological constant and can also be used to describe vacuum energy. However its introduction is unnecessary in the new model as it is better to to consider the curvature in its full generality for the purposes of cosmological application. For these purposes it is shown how to define local proper time in such a way that observer essentially see a homogeneous cosmology but with the inclusion of the modified curvature term. The Hubble constant and deceleration parameter are calculated in the usual way but now include a component

that reflects the generalised form of the curvature term. Hence when the observational data (an essentially flat universe coupled with a negative deceleration parameter) is introduced the Hubble constant and deceleration parameter are shown to provide information about the generalised nature of the curvature and the geometry of the modified cosmological model. In terms of this model the data is not interpreted in terms of dark energy or cold dark matter.

Finally the solution is discussed using a equation of state for an observer employing local proper time. Moreover the resulting solutions are also described in terms of problem of gravitational collapse. In particular it is shown how fluids including polytropes, with a central equation of state may be matched to Schwarzschild empty space-time.

References

[1] Kustaanheimo P and Qvist B 1948 *Comment. Phys. Math., Helsingf.* **13** 1

[2] McVittie G C 1933 *Mon. Not. R. Astron. Soc.* **93** 235

[3] Stephani H 1983 *J. Phys. A: Math. Gen.* **16** 3529

[4] Stephani H and Wolf T 1996 *Class. Quant. Grav.* **13** 1261

[5] Wiltshire R J 2006 *Class Quantum Grav* **23** 1365

[6] Nariai H 1967 *Prog. Theor. Phys* **38** 92

[7] Chakravarty N Choudhury S B D and Banerjee A. 1976 *Aust. J. Phy.* **29** 113

[8] McVittie, G C 1966 *Astrophys. J.* **143** 682

[9] McVittie G C 1984 *Ann. Inst. H. Poincaré* **3** 235

[10] Srivastava D C 1987 *Class. Quant. Grav.,* **4** 1093

[11] Sussman R A 1987 *Journal of Math. Phys.* **28** 1118

[12] Sussman R A 1988 *Journal of Math. Phys.* **29** 945

[13] Sussman R A 1988 *Journal of Math. Phys.* **29** 1177

[14] Knutsen, H *Int. J. Theor. Phys.* **25** 741

[15] Krasinski A 1997 *Inhomogeneous cosmological models* (Cambridge University Press, Cambridge)

[16] Stephani H, Kramer D, Maccallum M, Hoenselaers C. and Herlt E 2003 *Exact solutions of Einstein's field equations* (Cambridge University Press, Cambridge)

[17] Perlmutter, S et al 1997 *Astrophys J* **483** 565

[18] Riess AG 1998 *Astron J* 509 74

[19] Schmidt B P et al 1998 *Astrophys* J **507** 46

[20] Garnavich PM et al 1998 *Astrophys J* **509** 74

[21] Perlmutter, S et al 1999 *Astrophys J* **517** 565

[22] Turner MS and Riess AG 2001 arXiv:astro-ph/0106051v1

[23] Virey J M et al 2005 Phys Rev D **72** 061302

[24] Bonnor W B and Vickers P A 1981 *Gen. Rel. Grav.* **13** 29

[25] Oppenheimer J R and Synder H 1939 *Phys Rev* **56** 455

[26] Babak S V and Glampedakis K 2002 *Class Quantum Grav* **19** *2141*

[27] Kegeles L S 1978 *Phys Rev* D **18** 1020

[28] Misner C W and Sharp D H 1964 *Phys. Rev.* **136** 2B 571

[29] Cahill M E and McVittie G C 1970 *J. Math. Phys.* **11** 1382

[30] Perivolaropoulos L 2007 *Lecture Notes in Physics 720* (ed Lefteris Papantonopoulos Springer)

[31] Peacock J A 1999 *Cosmological Physics* (Cambridge University Press, Cambridge)

[32] Milne E A *Kinematical Relativity* (Clarenden Press)

[33] Weinberg S 1972 *Gravitation and Cosmology* (Wiley, New York)

[34] Wiltshire R J 2003 *Gen Rel & Grav* **35** 175

[35] WMAP 2006 *The Wilkinson Anisotropy Probe (WMAP)* http://map.gsfc.nasa.gov